2D CFT and the Conformal Bootstrap

David Grabovsky

February 20, 2023

Abstract

These notes are a review of some aspects and basic principles of 2-dimensional conformal field theory from the perspective of the conformal bootstrap. We discuss the Virasoro algebra and its representations, the constraints that conformal symmetry puts on CFT correlators, and how to use crossing symmetry to solve 2D CFTs.

Contents

1 Motivation and Bootstrap Propaganda

1.1 Intentionally Cryptic Prologue

You are lost at sea. Let $h(x)$ denote the height of the water above sea level at your location x on the Earth's surface. The heights $h(x)$ and $h(y)$ are highly correlated whenever x and y are so close together that they lie on the same wave. The average value of their product, $\langle h(x)h(y)\rangle$, is called a *correlator*: it is large when $|x-y|$ is small, and it falls to zero when $|x-y|$ is large. You can predict the correlator if you know something about either the dynamics of ocean waves or the symmetries underlying their shapes. The former involves tracking your boat's motion through the water and understanding how its height will change on average when you move from x to y: this is hard. The latter involves estimating $h(y)$ from $h(x)$ based on the average size, shape, and distribution of waves: this is easy.

1.2 The Bootstrap Approach

Definition 1.1 (CFT)**.** A **conformal field theory** (CFT) consists of a **spectrum** and a set of **correlators**, subject to certain symmetry assumptions and consistency conditions. To *define* a CFT is to give data that, at least in principle, uniquely specify its spectrum and correlators; to *solve* a CFT is to actually compute its spectrum and correlators.

The spectrum $\mathcal{S}$ is a set of numbers that characterizes the structure of the Hilbert space $\mathcal{H}_{\text{CFT}}$ of quantum states: we think of $\mathcal{S}$ as a set of energy levels. The states in $|\psi\rangle \in \mathcal{H}_{\text{CFT}}$ can, for now, be thought of as particles living on some surface. In QFT, quantum states live on constant-time slices of a spacetime, and in Euclidean signature these can be anything, including a sphere surrounding any point x. Since the theory is scale-invariant, we can shrink the sphere down to a point. In this way we associate to each state $|\psi\rangle$ an object $\mathcal{O}_\psi(x)$ at each x, and the collection of these objects as x varies is the **field** associated to $|\psi\rangle$. This is the **state-field correspondence**: it is one of the central tenets of CFT.

To each set of N fields $\mathcal{O}_i(x_i)$ we associate a number called their *N-point correlator*,

$$G_N(x_1, ..., x_N) = \left\langle \prod_{i=1}^{N} \mathcal{O}_i(x_i) \right\rangle \in \mathbb{C}. \tag{1.1}$$

These functions describe the degree of correlation between the fields $\mathcal{O}_i$; in other words, G_N is the probability amplitude for N particles to interact at points x_i. The notation above is purely formal, and is meant to convey only that correlators depend linearly on the fields. Strictly speaking, one does not need to define fields at all: there are only correlators. Nevertheless, it is useful to think of the $\mathcal{O}_i(x)$ as linear operators acting on $\mathcal{H}_{\text{CFT}}$, and of their correlators as vacuum expectation values: $G_N(x_1, ..., x_N) = \langle 0|\mathcal{O}_1(x_1)\cdots\mathcal{O}_N(x_N)|0\rangle$.

Symmetry. In any quantum theory, the Hilbert space $\mathcal{H}$ of states is a direct sum (counted with multiplicity) of irreducible unitary representations $\mathcal{R}$ of its symmetry algebra:

$$\mathcal{H} = \bigoplus_{\mathcal{R}} m_{\mathcal{R}} \mathcal{R}, \qquad m_{\mathcal{R}} \in \mathbb{N}, \qquad \mathcal{R} \text{ irreducible.} \tag{1.2}$$

(This is axiomatic: We specify the symmetries by choosing a Lie algebra, and we implement those symmetries by acting unitarily on the states in $\mathcal{H}$. This defines a unitary representation of the algebra, and all representations decompose into direct sums of irreducibles.)

In 2D CFT, conformal symmetry will be encoded in (two copies of) the Virasoro algebra. As we will see, conformal symmetry not only structures the Hilbert space and tells us about the spectrum, but also plays an important role in tightly constraining the correlators.

Consistency. We adopt two additional axioms to be satisfied by the correlators:

1. Correlators are commutative, e.g. $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \langle \mathcal{O}_2(x_2)\mathcal{O}_1(x_1)\rangle$.

2. There exists an **operator product expansion** (OPE):

$$\boxed{\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ij}^k(x_1, x_2)\mathcal{O}_k(x_2).} \tag{1.3}$$

   The complex-valued functions $C_{ij}^k(x_1, x_2)$ are called **OPE coefficients**, and they can be singular as $x_1 \longrightarrow x_2$. The sum over k runs over a basis of the CFT Hilbert space, and the sum is required to converge when x_1, x_2 are sufficiently close together.[1]

The OPE is extraordinarily powerful: it reduces N-point functions to sums of $(N-1)$-point functions. Since (as we shall show) conformal symmetry fully fixes all 2-point functions, repeated use of the OPE fully determines all of the correlators, up to the $C_{ij}^k(x_1, x_2)$. In fact, conformal symmetry also fixes the (x_1, x_2)-dependence of these functions, leaving only their normalization undetermined. So all that remains is to find the constants C_{ij}^k.

Now, the commutativity axiom implies that the OPE is commutative and associative. Through (1.3), these conditions encode equations that relate the OPE coefficients to each other. There are sometimes enough such relations to uniquely determine all of the C_{ij}^k, and therefore to solve the CFT. The art of actually doing this is the **conformal bootstrap**.

Advantages of the bootstrap. The bootstrap embraces an austere and elegant vision of what a CFT is. Only "experimental" data—energy levels and amplitudes—is necessary to define it, and the dream is that this data can be computed from the mere fact that the theory is self-consistent. No Lagrangian is required: one may well exist, but the bootstrap does not care about it. One does not even need physical fields, dynamics, a Hamiltonian, or

[1] As x_1 approaches x_2, we probe deeper into the UV. A CFT has no problem doing this, and the expansion is exact. But in ordinary QFTs, a UV cutoff prevents OPEs from being quite as precise or useful.

a phase space; moreover, path integrals and canonical quantization are unnecessary because the theory is already quantum. In practice, many of these structures often appear naturally and are even quite useful. The point is just that a 2D CFT does not rely, ontologically, on any of them: its heart and soul lies rather in its symmetries.

1.3 Outline of These Notes

In the next two sections, I will flesh out the theory whose skeleton has just haunted us above. And in the last section, I will show how to use the bootstrap to solve 2D CFTs.

- **Section 1:** Introduction, motivation, and the basic idea of the conformal bootstrap, including a sketch of the data that defines a CFT and the tools needed to solve it.
- **Section 2:** Conformal transformations in 2 dimensions, the Virasoro algebra, the formal structure of the 2D CFT Hilbert space, and several kinds of CFTs.
- **Section 3:** Radial quantization and the operator-state correspondence, conformal fields and their transformation properties, the stress tensor, and correlators.
- **Section 4:** How to use the OPE to decompose CFT correlators into conformal blocks; and how to use crossing symmetry to bootstrap your way to victory.

These notes are based heavily on Sylvain Ribault's beautiful review [1] and "minimal" lecture notes [2]. Other excellent sources include the Big Yellow Book [3] of di Francesco, Mathieu, and Senechal, and the lecture notes of Paul Ginsparg [4] and Xi Yin [5].

2 Representations of the Virasoro Algebra

2.1 Conformal Transformations

It is a standard fact from complex analysis that the only **conformal** (angle-preserving) **transformations** of the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are fractional-linear:

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0. \tag{2.1}$$

The group of these transformations is called the Möbius or **global conformal group** $\mathrm{PSL}(2,\mathbb{C})$. It is generated by translations, rotations, dilations, and inversions:

$$f_{\mathrm{tr}}(z) = z + a, \qquad f_{\mathrm{rot}}(z) = e^{i\theta} z, \qquad f_{\mathrm{dil}}(z) = cz, \qquad f_{\mathrm{inv}}(z) = \frac{1}{z}. \tag{2.2}$$

The global conformal group enlarges of the group of rigid transformations of $\mathbb{C}$ (that is, the translations and rotations) by adding, crucially, scale transformations, as well as maps that turn the complex plane "inside out." Here is a fun fact: $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SO}_0(3,1)$.

In fact, *every* holomorphic function $z \mapsto f(z)$ is angle-preserving. But if f is not Möbius, then it cannot be one-to-one and must have singularities: it is only *locally* conformal.[2] Locally, every meromorphic function admits a Laurent expansion $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, so it is linearly generated by the z^n. Accordingly, the infinite-dimensional algebra of local conformal transformations—called the **Witt algebra**—is infinitesimally generated by similar objects. Its generators, denoted by ℓ_n, satisfy the following commutation relations:

$$\ell_n = -z^{n+1}\frac{\partial}{\partial z} \implies [\ell_n, \ell_m] = (n-m)\ell_{n+m}, \qquad n \in \mathbb{Z}. \tag{2.3}$$

The subalgebra $\mathrm{PSL}(2,\mathbb{C})$ of global conformal transformations is generated by the three ℓ_n with $n \in \{0, \pm 1\}$. The rest of the generators describe local conformal transformations.

2.2 The Virasoro Algebra

We shall now upgrade the Witt algebra, which acts on the geometry, to the true symmetry algebra of the CFT, which acts on its Hilbert space. For subtle reasons arising from the fact that the Hilbert space is complex and projective, we need to *complexify* and *centrally extend* the Witt algebra.[3] The complexified Witt algebra has a complex basis $(\ell_n, \bar{\ell}_m)$ that consists of two identical and mutually commuting copies of the Witt algebra. The ℓ_n generate the "left-moving" or "chiral" or "holomorphic" conformal transformations, and the $\bar{\ell}_m$ generate the "right-moving" or "anti-chiral" or "anti-holomorphic" ones.[4]

The unique central extension of the Witt algebra is called the **Virasoro algebra**. Its generators, denoted by L_n, satisfy the following modified commutation relations:

$$\boxed{[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n\left(n^2-1\right)\delta_{n,-m},} \qquad c \in \mathbb{C}. \tag{2.4}$$

The constant c is called the **central charge**. It characterizes the CFT: roughly speaking, it measures the number of degrees of freedom in the theory. Note that c does not affect the generators of the global conformal group; and when $c = 0$, (2.4) is just the Witt algebra.

And so, to summarize: *the symmetry algebra* $\mathfrak{vir} \times \overline{\mathfrak{vir}}$ *of a 2D CFT is the direct product of two copies of the Virasoro algebra. It is generated by* $(L_n, \bar{L}_m)$ *for* $n, m \in \mathbb{Z}$*, where both the* L_n *and the* $\bar{L}_m$ *satisfy the algebra (2.4) in addition to* $[L_n, \bar{L}_m] = 0$.[5] *The Hilbert space* $\mathcal{H}_{\mathrm{CFT}}$ *is a direct sum of irreducible unitary representations of this algebra.*

[2] We can interpret a singularity at z_0 as indicating the nontrivial transformation of fields there. In the vacuum state $|0\rangle$, we can assume that a trivial field called the **identity field** $\mathbb{1} = \mathcal{O}_0$ lives at z_0, which becomes nontrivial after a local conformal transformation is applied. Thus, more generally, *global conformal transformations move fields around, while local conformal transformations also modify them.*

[3] See this StackExchange post for a beautiful explanation of why central extensions are necessary.

[4] Even though the complexified Witt algebra acts on the *complexified* complex plane $\mathbb{C}^2$, we will demand that the correlators continue to be functions on $\mathbb{C}$. Thus we will admit $\sqrt{|z|} = \sqrt{z\bar{z}}$, but not $\sqrt{z}$.

[5] If we assume invariance under the "time reversal" $z \mapsto -\bar{z}$, then the central charges agree: $c = \bar{c}$.

The global Virasoro generators have direct physical interpretations, and we postulate that they play important roles in the quantum theory we are about to develop:

- $L_0 + \overline{L}_0$ generates dilations. It is the Hamiltonian operator; it measures energy.
- $L_0 - \overline{L}_0$ generates rotations. It is the angular momentum; it measures spin.
- L_{-1} and $\overline{L}_{-1}$ generate translations in space. They are the left and right momenta.
- L_1 and $\overline{L}_1$ generate the so-called *special conformal transformations*, which are related to inversions. They play the role of the left and right boost operators.

In order to ensure that $L_0 + \overline{L}_0$ behaves like a Hamiltonian, we will require that both L_0 and $\overline{L}_0$ are (simultaneously) diagonalizable, and that $L_0 + \overline{L}_0$ is bounded from below. This interpretation suggests that the radial coordinate $|z|$ should play the role of time; indeed, this is the key idea behind radial quantization, which we will discuss below.

As we are about to see, the other Virasoro generators behave somewhat like the creation and annihilation operators familiar from the quantum harmonic oscillator. The L_{+n} are "lowering" operators, and we require that they all annihilate the vacuum state. Meanwhile, the L_{-n} are "raising" operators, and they create excitations with higher energy.[6]

In view of the fact that the symmetry algebra splits into two pieces, it is natural to assume (though it does not follow automatically!) that the irreducible representations in the CFT Hilbert space fall apart into left-moving and right-moving factors:

$$\mathcal{H}_{\text{CFT}} = \bigoplus_{R,\overline{R}'} m_{R,\overline{R}'} \big(R \otimes \overline{R}'\big), \qquad R, \overline{R}' \text{ irreducible.} \tag{2.5}$$

This leads to **holomorphic factorization**, the statement that quantities associated to irreducible representations $R \otimes \overline{R}'$ take the form $f(z)\overline{f}(\bar{z})$ for functions f, $\overline{f}$.

2.3 The CFT Hilbert Space

The operators L_0 and $\overline{L}_0$ are special. They commute and are simultaneously diagonalizable, and we can gain a detailed understanding of the CFT Hilbert space by describing a basis for $\mathcal{H}_{\text{CFT}}$ given by their eigenvectors. We have special names for their eigenvalues:

Operator	Eigenvalue	Terminology	What happens if it's zero?
$(L_0, \overline{L}_0)$	$(h, \overline{h})$	Conformal weights	$\|\psi\rangle$ is the vacuum state $\|0\rangle$
$H = L_0 + \overline{L}_0$	$\Delta = h + \overline{h}$	Conformal dimension	$\|\psi\rangle$ is the vacuum state $\|0\rangle$
$J = L_0 - \overline{L}_0$	$\ell = h - \overline{h}$	Angular momentum	$\|\psi\rangle$ is *scalar* or *diagonal*
$2\overline{L}_0$	$\tau = \Delta - \ell$	Twist	$\|\psi\rangle$ is *chiral* or *holomorphic*

[6] We have not yet defined an inner product on $\mathcal{H}_{\text{CFT}}$. When we do so, it will respect $L_n^\dagger = L_{-n}$.

Let $|\psi\rangle \in \mathcal{H}_{\text{CFT}}$ be an eigenvector of L_0 with weight h. Using the Virasoro algebra, we find that the state $L_n |\psi\rangle$ is also an L_0-eigenvector, with eigenvalue shifted by n:

$$L_0 |\psi\rangle = h |\psi\rangle \implies L_0\big(L_n |\psi\rangle\big) = (L_n L_0 - nL_n) |\psi\rangle = (h-n)\big(L_n |\psi\rangle\big). \tag{2.6}$$

This calculation shows that L_n lowers the energy of a state by n. Since the Hamiltonian is bounded from below, one cannot lower the energy indefinitely: there must exist an L_0-eigenstate $|\psi\rangle$ that is annihilated by L_n for all $n > 0$. Any state $|\psi\rangle \in \mathcal{H}_{\text{CFT}}$ for which

$$L_0 |\psi\rangle = h |\psi\rangle, \quad \overline{L}_0 |\psi\rangle = \bar{h} |\psi\rangle, \quad \text{and} \quad L_n |\psi\rangle = \overline{L}_n |\psi\rangle = 0 \quad \text{for all } n > 0 \tag{2.7}$$

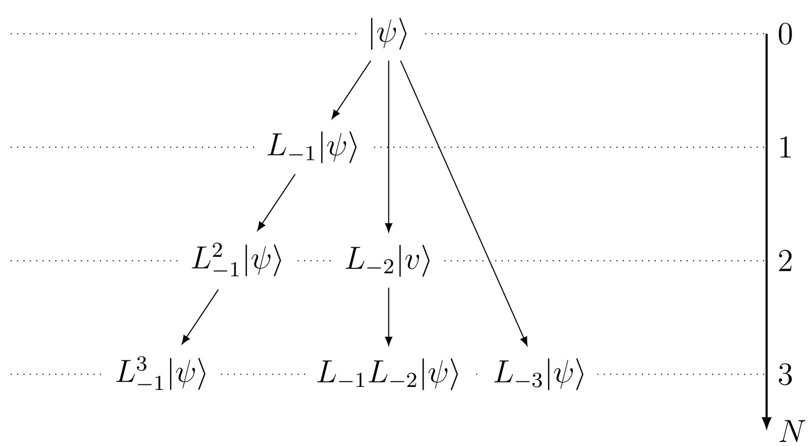
is called a **primary** or **highest-weight** state. (Somewhat confusingly, primaries are the states of *lowest* energy—lowest conformal weight—in a given irreducible representation.)

The calculation (2.6) also shows that one can *increase* the weight of a primary state $|\psi\rangle$ by acting on it with an arbitrary string of Virasoro generators L_{n_i} with $n_i < 0$. We can always (WLOG) write any such state as a linear combination of states of the form

$$L_{-n_1} \cdots L_{-n_k} |\psi\rangle, \qquad 0 \le n_1 \le \cdots \le n_k \tag{2.8}$$

by using the Virasoro algebra to commute the generators past each other. All such states are L_0-eigenvectors; their weights are $h + \sum_{i=1}^k n_i = h + N$, where the integer N is called the **level**. States with level $N \ge 1$ are called **descendants**, and the vector space generated by linear combinations of the descendants (2.8) is called the **Verma module** $\mathcal{V}_h$.

Starting with each primary state $|\psi\rangle$, an infinite tower of descendants fans out. The number of independent states at level N is $p(N)$, the number of integer partitions of N.



Each Verma module $\mathcal{V}_h$ furnishes a representation of the Virasoro algebra, but such a representation may fail to be irreducible: it may contain nontrivial subrepresentations, and these must be quotiented out before we proceed. To see this more explicitly, we endow $\mathcal{V}_h$

with an inner product in accord with the requirement $L_n^\dagger = L_{-n}$:

$$\Big(L_{-n_1}\cdots L_{-n_r}\left|\psi\right\rangle, \, L_{-m_1}\cdots L_{-m_s}\left|\psi\right\rangle\Big) = \left\langle\psi\right|L_{n_r}\cdots L_{n_1}L_{-m_1}L_{-m_s}\left|\psi\right\rangle, \tag{2.9}$$

where the bras of primary states satisfy $\left\langle\psi\right| L_0 = \left\langle\psi\right| h$ and $\left\langle\psi\right| L_n = 0$ for all $n < 0$. In this inner product, descendants of different levels are automatically orthogonal.

If $\mathcal{V}_h$ has a nontrivial subrepresentation, then it contains a full-fledged submodule $\mathcal{V}_\chi$ generated by a primary state $\left|\chi\right\rangle$ that is, at the same time, a descendant of $\left|\psi\right\rangle$. (This only happens when there is some definite relation between c and h.) Primaries that are also descendants are called **null vectors**, because they have zero norm: $\langle\chi|\chi\rangle = 0$.[7] Moreover, every descendant of $\left|\chi\right\rangle$ has vanishing norm as well, so the whole Verma module $\mathcal{V}_\chi$ is orthogonal to the rest of $\mathcal{V}_h$. The quotient $R_h = \mathcal{V}_h/\mathcal{V}_\chi$ gets rid of these null states by identifying any two states in $\mathcal{V}_h$ that differ by a null vector in $\mathcal{V}_\chi$, and the inner product induced on R_h from (2.9) is positive-definite. What remains, after all necessary quotients have been performed, is a bona fide irreducible representation of the Virasoro algebra.

Let us summarize what we have learned about the CFT Hilbert space. Each primary state $\left|\psi\right\rangle \in \mathcal{H}_{\mathrm{CFT}}$, labeled by its weights $(h,\bar{h})$, generates a Verma module $\mathcal{V}_{h,\bar{h}} = \mathcal{V}_h \otimes \overline{\mathcal{V}}'_{\bar{h}}$ by the action of strings of Virasoro generators L_{-n_i}. If the modules $\mathcal{V}_h$ or $\overline{\mathcal{V}}'_{\bar{h}}$ are reducible, we find all of their null vectors and quotient out the submodules they generate, leaving us with irreducible representations R_h and $\overline{R}'_{\bar{h}}$. The Hilbert space $\mathcal{H}_{\mathrm{CFT}}$ is then a direct sum, running over every primary state in the CFT, of these irreducible representations:

$$\boxed{\mathcal{H}_{\mathrm{CFT}} = \bigoplus_{h,\bar{h}} m_{h,\bar{h}}\big(R_h \otimes \overline{R}'_{\bar{h}}\big),} \qquad (h,\bar{h}) \in \mathcal{S}. \tag{2.10}$$

The **spectrum** of the CFT is the list $\mathcal{S}$ of the conformal weights of all primaries in the theory. The theory may contain either finitely or infinitely many primaries; either way, the data in $\mathcal{S}$ is the minimal information necessary to reconstruct the Hilbert space.

2.4 Various Kinds of CFTs

Let us close this section by distinguishing various classes of conformal field theories:

- A CFT is **unitary** if $\mathcal{H}_{\mathrm{CFT}}$ has a positive-definite inner product where $L_0 + \overline{L}_0$ is self-adjoint. Only in unitary CFTs are all states normalizable, and only these CFTs are physical. Every unitary CFT has $c \geq 0$ and all weights $h, \bar{h} \geq 0$. (Unitarity was assumed in our discussion above, and indeed we will consider only unitary CFTs.)

- A CFT is **compact** if its spectrum is real and discrete, and there if is a unique state with $\Delta = 0$. This state is the vacuum $\left|0\right\rangle$, and its associated field is the identity $\mathbb{1}$.

[7]This does *not* signal a failure of certain descendants at some level to be linearly independent: $\left|\chi\right\rangle$ is *not* the zero vector. It is rather a failure of the "inner product" defined above to be positive-definite.

The vacuum is $\mathrm{PSL}(2,\mathbb{C})$-invariant, meaning that $L_n|0\rangle = 0$ for all $n \geq -1$.

- A CFT is **rational** if it has a finite number of primary states. It can be shown that in rational CFTs, all conformal dimensions (as well as c) must be rational numbers.
- A rational CFT is a **minimal model** if it satisfies a few other assumptions. There is an ADE classification of the minimal models, and they have all been solved exactly.

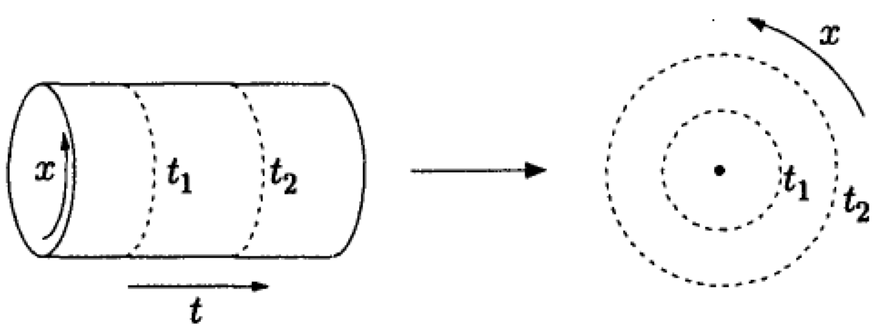
My favorite minimal model is the **trivial CFT**, which has $c = \bar{c} = 0$[8] and whose spectrum consists solely of the vacuum $|0\rangle$, which has $h = \bar{h} = 0$. The CFT Hilbert space is a single irreducible representation built from the vacuum and its Virasoro descendants. Unitarity (i.e. getting rid of the null states) forces this to be the trivial representation, so in fact there is only one physical state. The complete list of all correlators is $\left\langle \prod_{i=1}^{N} \mathbb{1}(z_i) \right\rangle = 1$.

A few nontrivial examples of 2D CFTs include the free boson (real, complex, compact); the free fermion (ditto, with various choices of boundary conditions); minimal models such as the Ising, Lee–Yang, and Potts models; Liouville theory; WZW models; holographic CFTs; and more. We will not discuss any of these, except for some AdS/CFT at the end.

3 Conformal Fields and Correlators

3.1 Radial Quantization

The theory we have been describing lives on the flat Minkowski plane, where time runs upward and the spatial direction is horizontal. But the infinite extent of space will produce IR divergences in the theory, and the theory is easier to formulate in Euclidean signature. Therefore we Wick rotate the metric and compactify the spatial direction: the result is the (Euclidean) cylinder $\mathbb{R} \times S^1$. The cylinder can be parametrized by the complex coordinate $\zeta = t + ix$, where $x \sim x + L$, and the conformal transformation $z = e^{2\pi\zeta/L}$ "explodes" the cylinder onto the plane. On the z-plane, time runs radially outward: the far past ($t \longrightarrow -\infty$) is at the origin ($z = 0$), and the far future ($t \longrightarrow +\infty$) is at infinity ($z = \infty$).



[8]CFTs with $c = 0$ arise in string theory, where conformal symmetry appears as a gauge symmetry. One requires $c = 0$ to cancel a Weyl anomaly, and this ultimately fixes the number of spacetime dimensions.

The idea of **radial quantization** is that, after performing these manipulations, the Hamiltonian—which generates time translations—corresponds to the dilation operator. But as we have seen, this is precisely what $H = L_0 + \overline{L}_0$ does. Thus radial quantization is what motivates our earlier assumptions on the boundedness and self-adjointness of H, and explains why we wanted to treat scaling eigenvalues as energies. Moreover, the fact that the constant-time slices are now concentric circles makes our original argument in favor of the state-field correspondence more concrete. We often say that the entire CFT Hilbert space lives at the origin: this is the basic fact that allows CFT states to define fields.

N.B. In QFT, fields are *operator-valued* distributions on spacetime, so it is often useful to think of the map $|\psi\rangle \longleftrightarrow \mathcal{O}_\psi(z,\bar{z})$ as an **operator-state correspondence**. Note that in any QFT with a vacuum state $|0\rangle$, any operator $\mathcal{O}_\psi$ defines a state by creating it from the vacuum: $|\psi\rangle \equiv \mathcal{O}_\psi(0)\,|0\rangle$. It is only in CFT that the map goes the other way too.

3.2 Fields and the Stress Tensor

By the operator-state correspondence, we allow the Virasoro algebra to act on fields as well as on states, and we distinguish between **primary** and **descendant fields**, which correspond to primary and descendant states, respectively. A primary field $\mathcal{O}_{h,\bar{h}}$ obeys

$$L_0\mathcal{O}_{h,\bar{h}} = h\mathcal{O}_{h,\bar{h}}, \quad \overline{L}_0\mathcal{O}_{h,\bar{h}} = \bar{h}\mathcal{O}_{h,\bar{h}}, \quad \text{and} \quad L_n\mathcal{O}_{h,\bar{h}} = \overline{L}_n\mathcal{O}_{h,\bar{h}} = 0 \quad \text{for all } n > 0. \tag{3.1}$$

It is possible to show, using the tools we are about to introduce, that primary fields transform covariantly under any local conformal transformation $z \mapsto w(z)$, as follows:

$$\boxed{\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \longmapsto \mathcal{O}_{h,\bar{h}}(w,\overline{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h}\left(\frac{\mathrm{d}\overline{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{h}}\mathcal{O}_{h,\bar{h}}(z,\bar{z}).} \tag{3.2}$$

Fields that satisfy (3.2) only for *global* conformal transformations are called **quasi-primary**, and their associated states are primary with respect to the global algebra $\mathrm{PSL}(2,\mathbb{C})$.

We wish to understand the action of the Virasoro algebra on the fields: that is, we want to describe the z-dependence of $L_n\mathcal{O}_{|\psi\rangle}(z) = \mathcal{O}_{L_n|\psi\rangle}(z)$. In accordance with our interpretation of L_{-1} and $\overline{L}_{-1}$ as the generators of translations, we postulate that for any field $\mathcal{O}$,

$$L_{-1}\mathcal{O}(z,\bar{z}) = \frac{\partial}{\partial z}\mathcal{O}(z,\bar{z}), \qquad \overline{L}_{-1}\mathcal{O}(z,\bar{z}) = \frac{\partial}{\partial \bar{z}}\mathcal{O}(z,\bar{z}). \tag{3.3}$$

We can then work out the z-dependence of L_n by applying (3.3) to the field $L_n\mathcal{O}$:

$$\frac{\partial}{\partial z}\big(L_n\mathcal{O}(z,\bar{z})\big) = L_{-1}L_n\mathcal{O}(z,\bar{z}) = -(n+1)L_{n-1}\mathcal{O}(z,\bar{z}) \implies \frac{\partial L_n}{\partial z} = -(n+1)L_{n-1}. \tag{3.4}$$

These relations show how the basis of Virasoro generators, in their action on fields, changes from point to point. If we define the **stress tensor** or **energy-momentum tensor** by

$$\boxed{T(w) = \sum_{n\in\mathbb{Z}} \frac{L_n}{(w-z)^{n+2}} \iff L_n = \frac{1}{2\pi i}\oint \mathrm{d}w\,(w-z)^{n+1}T(w),} \tag{3.5}$$

then all of the relations (3.4) are encapsulated in the equations $\dfrac{\partial}{\partial z}T(w) = \dfrac{\partial}{\partial \bar{z}}T(w) = 0$.

Thus we have the answer to our question about how $L_n\mathcal{O}$ depends on z:

$$T(w)\mathcal{O}(z) = \sum_{n\in\mathbb{Z}} \frac{(L_n\mathcal{O})(z)}{(w-z)^{n+2}} \iff (L_n\mathcal{O})(z) = \frac{1}{2\pi i}\oint \mathrm{d}w\,(w-z)^{n+1}T(w)\mathcal{O}(z). \tag{3.6}$$

When $\mathcal{O}$ is primary, it is annihilated by the positive Virasoro modes. In this case, (3.6) tells us—by Cauchy's integral formula—that descendant fields are essentially derivatives of primary fields: $L_n\mathcal{O} \sim \partial^n\mathcal{O}$. Using (3.3) and the definition of a primary field, we find

$$\boxed{T(w)\mathcal{O}(z) = \frac{h\mathcal{O}(z)}{(w-z)^2} + \frac{\partial\mathcal{O}(z)}{(w-z)} + (\text{regular}).} \qquad (\mathcal{O}\ \text{ primary}) \tag{3.7}$$

This is our first example of an operator product expansion (OPE). Here is another one; it follows directly from the mode expansion (3.6) and by using the Virasoro algebra:

$$\boxed{T(w)T(z) = \frac{(c/2)\mathbb{1}}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{(w-z)} + (\text{regular}).} \tag{3.8}$$

By comparing this expansion to the OPE of T with a primary, we see that T is *not* primary. It is, however, quasi-primary, and is in fact a descendant at level 2 of the identity field: $T(z) = L_{-2}\mathbb{1}(z)$. It has dimension and spin 2: $(h,\bar{h}) = (2,0) \implies \Delta = \ell = 2$.

The stress tensor is perhaps the single most important object in 2D CFT. Its definition (3.5) can be viewed as a Laurent expansion in w, convergent as z approaches w, whose modes are the Virasoro generators.[9] A few other important facts about it follow:

- $T(z)$ combines with its anti-chiral twin $\overline{T}(\bar{z})$ to form a 2×2 traceless, symmetric, divergence-free matrix $T_{\mu\nu}$ with nonzero entries $T_{zz} = T(z)$ and $T_{\bar{z}\bar{z}} = \overline{T}(\bar{z})$. It is the Noether current for Virasoro symmetry, and the "conformal charge" is

$$Q[\varepsilon(z,\bar{z})] = \frac{1}{2\pi i}\oint \mathrm{d}z\, T(z)\varepsilon(z) + \frac{1}{2\pi i}\oint \mathrm{d}\bar{z}\,\overline{T}(\bar{z})\bar{\varepsilon}(\bar{z}). \tag{3.9}$$

These charges generalize the angular momenta and boosts of the Lorentz group.

[9] We chose to use the Virasoro algebra to construct the stress tensor, but we could have begun by assuming the existence of a stress tensor with certain properties, and shown that the modes L_n in (3.5) satisfy the Virasoro algebra. The point is that $T(w)$ knows everything about conformal symmetry in a 2D CFT.

- Under conformal transformations $z \mapsto w(z)$, the stress tensor transforms to

$$T(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2}\left[T(z) - \frac{c}{12}\{w, z\}\right], \qquad \{w, z\} = \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2. \tag{3.10}$$

  The bizarre-looking term $\{w, z\}$ is a *Schwarzian derivative.* Since it is proportional to c, one may view it as a quantum correction. Indeed, $\{w, z\}$ vanishes when $w(z)$ is a Möbius transformation, so it measures the failure of $w(z)$ to be globally conformal.

- Consider the transformation (3.10) with $w(z) = -\frac{1}{z}$. One can check that $\{w, z\} = 0$, so we find that $T(-1/z) = z^{-4}T(z)$. If we demand that $T(0)$ must be regular, then we see that $T(z)$ is not only holomorphic at infinity, but decays at large z as z^{-4}.

3.3 Correlation Functions

We are now ready to discuss the central objects of 2D CFT. To N fields $\mathcal{O}_1(z_1), ..., \mathcal{O}_N(z_N)$ and distinct points $z_1 \neq \cdots \neq z_N$, we associate a number called the **correlator** or N**-point function** $\langle\mathcal{O}_1(z_1)\cdots\mathcal{O}_N(z_N)\rangle$. Correlators must be single-valued and meromorphic in c, the weights $(h_i, \bar{h}_i)$, and the positions z_i. We assume that fields commute inside correlators, so $\langle\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\rangle = \langle\mathcal{O}_2(z_2)\mathcal{O}_1(z_1)\rangle$. Correlators are also linear in the fields, so that

$$\frac{\partial}{\partial z_1}\big\langle\mathcal{O}_1(z_1)\cdots\mathcal{O}_N(z_N)\big\rangle = \left\langle\frac{\partial}{\partial z_1}\mathcal{O}_1(z_1)\cdots\mathcal{O}_N(z_N)\right\rangle. \tag{3.11}$$

By linearity and because descendants can be built from primaries, it is enough to compute the correlators of primary fields. To understand and constrain their behavior, we will insert the stress tensor into the correlator. For a string of N primaries $\mathcal{O}_i(z_i)$, define

$$G_N = \left\langle\prod_{i=1}^N \mathcal{O}_i(z_i)\right\rangle, \qquad G_N(w) = \left\langle T(w)\prod_{i=1}^N \mathcal{O}_i(z_i)\right\rangle. \tag{3.12}$$

By the $T\mathcal{O}$ OPE (3.7), the effect of inserting $T(w)$ is to apply a differential operator:

$$G_N(w) = \sum_{i=1}^N \left[\frac{h_i}{(w - z_i)^2} + \frac{1}{(w - z_i)}\frac{\partial}{\partial z_i}\right]G_N. \tag{3.13}$$

Now recall that at large w, we have $T(w) \sim w^{-4}$. It follows that the coefficients at w^{-1}, w^{-2}, and w^{-3} in the expansion of $G_N(z_i, w)$ around $w = \infty$ must all vanish. In this way we obtain three differential equations, known as the **Ward identities**, satisfied by the correlator:

$$\boxed{\sum_{i=1}^N \frac{\partial}{\partial z_i}G_N = \sum_{i=1}^N \left(z_i\frac{\partial}{\partial z_i} + h_i\right)G_N = \sum_{i=1}^N \left(z_i^2\frac{\partial}{\partial z_i} + 2h_i z_i\right)G_N = 0.} \tag{3.14}$$

As examples, let us solve the Ward identities for a correlator of $N = 0, 1, 2, 3$ primaries.

- $N = 0$: The Ward identities read $0 = 0$. The zero-point function $\langle \cdot \rangle$, sometimes called the sphere partition function, must be a constant which we may set to $\langle \cdot \rangle = 1$.
- $N = 1$: The Ward identities are overdetermined; they give

$$\frac{\partial}{\partial z}\langle \mathcal{O}(z)\rangle = h\langle \mathcal{O}(z)\rangle = 0. \tag{3.15}$$

  Thus either $\langle \mathcal{O}(z)\rangle = 0$, or else $h = 0$ and $\mathcal{O}$ is the identity field with $\langle \mathbb{1}(z)\rangle = 1$.
- $N = 2$: There are still more Ward identities than unknowns. We will find a unique solution, plus a constraint on h_1 and h_2. The Ward identities are equivalent to

$$\begin{aligned} (z_1 - z_2)(h_1 - h_2)\big\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\big\rangle &= 0, \\ \frac{\partial}{\partial z_1}\big\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\big\rangle = -\frac{\partial}{\partial z_2}\big\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\big\rangle &= -\frac{2h_1}{z_1 - z_2}\big\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\big\rangle. \end{aligned} \tag{3.16}$$

  The first equation shows that for $z_1 \neq z_2$, $\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\rangle = 0$ unless $h_1 = h_2$. The second equation has solution $\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\rangle = A(z_1 - z_2)^{-2h}$, with A a constant that we often set to 1. The same story holds anti-holomorphically, so in summary

$$\boxed{\big\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\big\rangle = \frac{\delta_{12}}{|z_1 - z_2|^{-2\Delta}},} \qquad \delta_{12} = \delta_{h_1 h_2}\delta_{\bar{h}_1 \bar{h}_2}. \tag{3.17}$$

  The absolute value notation introduced above is peculiar to 2D CFT: it is meant to be read "$|f(z)| = f(z)\overline{f}(\bar{z})$," in accordance with holomorphic factorization.
- $N = 3$: We have as many equations as unknowns, so we will find a unique solution for $G_3(z_1, z_2, z_3)$ with no constraints on the h_i. After some work, one obtains

$$\boxed{\big\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\big\rangle = C_{123}|z_{12}|^{\Delta_3-\Delta_1-\Delta_2}|z_{13}|^{\Delta_2-\Delta_1-\Delta_3}|z_{23}|^{\Delta_1-\Delta_2-\Delta_3},} \tag{3.18}$$

  where we have introduced the shorthand $z_{ij} = z_i - z_j$. The constant C_{123} cannot be set to 1, because the OPE reduces this correlator to a sum of two-point functions that have already been normalized. As we will see, the C_{ijk} are really OPE coefficients.

For $N \geq 4$, we run out of luck: there are more unknowns than Ward identities to constrain them. For instance, for $N = 4$, the general solution to the Ward identities is

$$\begin{aligned} G_4(z_1, z_2, z_3, z_4) &= z_{12}^{-2\Delta_1} z_{23}^{\Delta_1-\Delta_2-\Delta_3+\Delta_4} z_{24}^{-\Delta_1-\Delta_2+\Delta_3-\Delta_4} z_{34}^{\Delta_1+\Delta_2-\Delta_3-\Delta_4} F_{1234}(z), \\ F_{1234}(z) &= \big\langle \mathcal{O}_1(z)\mathcal{O}_2(0)\mathcal{O}_3(\infty)\mathcal{O}_4(1)\big\rangle, \qquad z = \frac{z_{12}z_{34}}{z_{13}z_{24}}. \end{aligned} \tag{3.19}$$

Here z is called the **conformal cross-ratio**, and F (whose form can be deduced from the transformation law (3.2)) is a function that the Ward identities cannot determine. More generally, for an N-point function, the Ward identities leave unfixed a function of $N-3$ cross-ratios. The space of solutions to the Ward identities is quite interesting: we will now describe a particular basis of their solutions whose elements we call *conformal blocks*.

4 The OPE and Crossing Symmetry

4.1 The Operator Product Expansion

Our main tool for computing correlators is the OPE, introduced as an axiom in (1.3) and exemplified in calculations in (3.7) and (3.8). There are three main ways to understand it:

1. **As a reflection of Wick's theorem:** In QFT, Wick's theorem relates different operator product orderings. In radial quantization, we often start with *radial* ("time") ordering and want to convert to normal-ordered products, which are free of divergences. The singular part of the OPE of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ is precisely the difference $\mathcal{R}\{\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\} - {:}\mathcal{O}_1(z_1)\mathcal{O}_2(z_2){:}$ between the two orderings of the products.[10]

2. **As a resolution of the identity:** The OPE is a consequence of the state-field correspondence and the identity $\mathbb{1} = \sum_\psi |\psi\rangle\langle\psi|$, where ψ runs over a basis of $\mathcal{H}_{\text{CFT}}$:

$$\mathcal{O}_i(z_1)\mathcal{O}_j(z_2) = \sum_\psi \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\,|\psi\rangle\langle\psi| = \sum_\psi \langle\psi|\mathcal{O}_1\mathcal{O}_2\rangle\,\mathcal{O}_\psi(z_2). \tag{4.1}$$

   Here we have replaced the field $\mathcal{O}_1\mathcal{O}_2$ by its corresponding state, and the state $|\psi\rangle$ with its corresponding field: note the resemblance to (1.3)! We often call $\mathcal{O}_\psi$ the "internal" or "exchanged" operator; the terminology will become clear soon.

3. **As a Laurent expansion for CFT operators:** If we apply the Ward identities (or insert T) to both sides of the OPE (1.3), it can be shown that in fact

$$C^k_{ij}(z_1, z_2) = C^k_{ij}|z_1 - z_2|^{\Delta_k - \Delta_i - \Delta_j}. \tag{4.2}$$

   Organizing the OPE by the contribution of each primary in the spectrum, we write

$$\boxed{\mathcal{O}_i(z_1)\mathcal{O}_j(z_2) = \sum_{k \text{ primary}} C^k_{ij}|z_1 - z_2|^{\Delta_k - \Delta_i - \Delta_j}\Big(\mathcal{O}_k(z_2) + O(z_1 - z_2)\Big).} \tag{4.3}$$

   where we have absorbed the contributions of descendants into the term $O(z_1 - z_2)$. In this way the coefficient at each primary $\mathcal{O}_k$ entering the OPE of $\mathcal{O}_i$ and $\mathcal{O}_j$ is either

[10] The necessity of ordering originates in Lorentzian signature. In Euclidean signature, one is free to choose any point as the origin for radial quantization, and in this way one obtains different radial orderings.

suppressed or made more singular by powers of z_{12}. Moreover, one can show that C_{ij}^k is precisely the 3-point normalization constant C_{ijk} by computing $\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\mathcal{O}_k(z_3)\rangle$ using the $\mathcal{O}_i\mathcal{O}_j$ OPE and comparing this with the known result (3.18).

And so, at long last: *The minimal input data that uniquely determines a CFT consists of the spectrum $\mathcal{S} = \{(h_i, \bar{h}_i)\}$ of primary weights; the central charges $c, \bar{c}$; and the OPE coefficients C_{ij}^k.* Given this data, one can use the OPE to compute all of its correlators.

4.2 The Crossing Equation

Let us use the OPE to continue the calculation (3.19) of the four-point function. The $\mathcal{O}_1\mathcal{O}_2$ OPE gives an expansion, organized again by primaries, that converges in the disk $|z| < 1$:

$$\begin{aligned} \Big\langle \mathcal{O}_1(z)\mathcal{O}_2(0)\mathcal{O}_3(\infty)\mathcal{O}_4(1)\Big\rangle &= \sum_{\Delta\in\mathcal{S}} C_{12\Delta}|z|^{\Delta-\Delta_1-\Delta_2}\Big(\Big\langle \mathcal{O}_\Delta(z)\mathcal{O}_3(\infty)\mathcal{O}_4(1)\Big\rangle + O(|z|)\Big) = \\ &= \sum_{\Delta\in\mathcal{S}} C_{12\Delta}C_{\Delta 34}|z|^{\Delta-\Delta_1-\Delta_2}\Big(1+O(|z|)\Big). \end{aligned} \tag{4.4}$$

The function $1 + O(|z|)$ above captures the OPE contributions of all descendants of $\mathcal{O}_\Delta(z)$. The contributions of the L_{-n} and $\bar{L}_{-n}$ holomorphically factorize, so we can write it as $\mathcal{F}_h^{(s)}(z)\mathcal{F}_{\bar{h}}^{(s)}(\bar{z})$, where (s) stands for “s-channel.” The **Virasoro conformal block** $\mathcal{F}_h^{(s)}(z)$ is the OPE contribution of the Verma module $\mathcal{V}_h$ to the four-point function $F_{1234}(z)$.[11] It is analytic in z, c, h_i, and h, and (in principle) entirely determined by conformal symmetry.[12]

The associativity of the OPE allows us to fuse together any two fields in the four-point function. For example, the $\mathcal{O}_1\mathcal{O}_4$ OPE gives an expansion convergent in $|1-z| < 1$:

$$\Big\langle \mathcal{O}_1(z)\mathcal{O}_2(0)\mathcal{O}_3(\infty)\mathcal{O}_4(1)\Big\rangle = \sum_{\Delta\in\mathcal{S}} C_{\Delta 14}C_{23\Delta}|1-z|^{\Delta-\Delta_1-\Delta_4}\Big(1+O(|1-z|)\Big). \tag{4.6}$$

The function $1 + O(|1-z|) = \mathcal{F}_h^{(t)}(z)\mathcal{F}_{\bar{h}}^{(t)}(\bar{z})$ factorizes as well; the subscript (t) stands for “t-channel.” Of course, by the commutativity of fields inside correlators, the $\mathcal{O}_1\mathcal{O}_4$ OPE could have been an $\mathcal{O}_1\mathcal{O}_2$ OPE if we had relabeled the fields and their insertion points. It follows that the s-channel blocks and the t-channel blocks are related by

$$\mathcal{F}_h^{(t)}(c, h_1, h_2, h_3, h_4; z) = \mathcal{F}_h^{(s)}(c, h_1, h_4, h_3, h_2; 1-z).$$

[11]Warning: not everyone defines the Virasoro blocks the same way! It is also common to normalize the blocks so that $\mathcal{F}_h^{(s)}(z) = z^{h-h_1-h_2}\big(1+O(z)\big)$, so as to make their behavior under scaling more manifest.

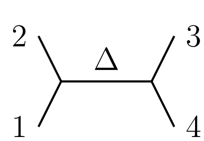
[12]Actually, the blocks are only *locally* analytic in z: they can have nontrivial monodromies when fields are moved around each other. For instance, permuting $\mathcal{O}_1$ and $\mathcal{O}_2$ has the following effect:

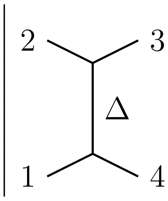
$$\mathcal{F}_h^{(s)}(c, h_1, h_2, h_3, h_4; z) = e^{i\pi(h-h_1-h_2)}(1-z)^{-h_1-h_2+h_3-h_4}\mathcal{F}_h^{(s)}\left(c, h_2, h_1, h_3, h_4; \frac{z}{z-1}\right). \tag{4.5}$$

And now comes the crucial insight: The s-channel and t-channel expansions (4.4) and (4.5) must agree with each other where they both converge. We therefore obtain

$$\begin{aligned}\big\langle \mathcal{O}_1(z)\mathcal{O}_2(0)\mathcal{O}_3(\infty)\mathcal{O}_4(1)\big\rangle &= \sum_{\Delta\in\mathcal{S}} C_{12\Delta}C_{\Delta 34}|z|^{\Delta-\Delta_1-\Delta_2}\mathcal{F}^{(s)}_h(z)\mathcal{F}^{(s)}_{\bar h}(\bar z) = \\ &= \sum_{\Delta\in\mathcal{S}} C_{\Delta 14}C_{23\Delta}|1-z|^{\Delta-\Delta_1-\Delta_4}\mathcal{F}^{(t)}_h(z)\mathcal{F}^{(t)}_{\bar h}(\bar z).\end{aligned} \tag{4.7}$$

This is the **crossing equation**, and the demand that all CFT correlators satisfy it is called **crossing symmetry**. Crossing encodes the associativity of the OPE, and can be represented by the following diagram showing how the OPE "contracts" the operators:

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = \sum_{\Delta\in\mathcal{S}} C_{12\Delta}C_{\Delta 34}\left|\right.$$


$$\left.\right|^2 = \sum_{\Delta\in\mathcal{S}} C_{23t}C_{t41}\left|\right.$$


$$\left.\right|^2 . \tag{4.8}$$

The similarity of (4.8) to Feynman diagrams justifies the terminology of "internal" operators, "external" dimensions, and so on. The notion that we are studying scattering amplitudes explains why conformal blocks are sometimes called **conformal partial waves**, in analogy to how spherical harmonics form the building blocks of the theory of atomic orbitals.

Regarding the conformal blocks as known objects, the crossing equation gives an infinite set of relations that are *quadratic* in the OPE coefficients. Their non-linearity in the C_{ijk} is part of what makes the conformal bootstrap so hard. The blocks themselves are also hard to compute, because they typically involve infinite sums. There are, however, efficient recursion relations due to Zamolodchikov that can be viewed as series expansions of the blocks, either in large c or in large Δ, that take advantage of their analytic structure in c and Δ.

4.3 An Example from AdS/CFT

We conclude with an example close to my heart. Suppose that c is very large, and consider the "heavy-light" (HHLL) four-point function of two scalar primaries $\mathcal{O}_L$ and $\mathcal{O}_H$:

$$G_\alpha(z) = \big\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\big\rangle, \qquad \Delta_L \ll \Delta_H = \frac{c}{12}\big(1-\alpha^2\big). \tag{4.9}$$

This correlator models the two-point function of a light probe field in a heavy background state $|\mathcal{O}_H\rangle$. In AdS_3/CFT_2, $\mathcal{O}_H$ represents a heavy object in AdS[13] that causes the bulk geometry to backract, while $\mathcal{O}_L$ represents a free scalar field propagating on the background created by $\mathcal{O}_H$. We might be interested in how $\mathcal{O}_H$ affects the physics of light probes; or

[13] This could be either a conical defect or a BTZ black hole, depending on the value of the "thermal" parameter α. If $\alpha \in (0,1)$, then $\Delta_H < \frac{c}{12}$ is a defect, while if $\alpha \in i\mathbb{R}$, then $\Delta_H > \frac{c}{12}$ is a black hole.

more precisely, what effect Δ_H has on the behavior of $G_\alpha(z)$ near $z = 1$.

The OPE is uniquely suited to such questions. Let's look at the t-channel OPE, which brings the two light operators together and probes the four-point function near $z = 1$:

$$G_\alpha(z) = \sum_{\Delta \in \mathcal{S}} C_{\Delta LL} C_{HH\Delta} |1-z|^{\Delta - 2\Delta_L} \mathcal{F}_h^{(t)} \mathcal{F}_{\bar{h}}^{(t)} = |1-z|^{-2\Delta_L} \left| \mathcal{F}_0^{(t)}(z) \right|^2 + \cdots . \tag{4.10}$$

The t-channel blocks $\mathcal{F}_h^{(t)}(z)$ are analytic at $z = 1$, so the most singular contribution to the OPE is carried by the primary with the smallest value of Δ—the identity field. (This phenomenon is known as **vacuum block dominance**.) In fact the t-channel blocks are known explicitly in the HHLL limit $c \longrightarrow \infty$ with $h_L/c \longrightarrow 0$ and h_H/c fixed:

$$\mathcal{F}_h^{(t)}(z) \approx z^{(\alpha-1)h_L} (1-z)^{2h_L} \left(\frac{1-z^\alpha}{\alpha} \right)^{h-2h_L} {}_2F_1\big(h, h; 2h; 1-z^\alpha\big). \tag{4.11}$$

Here ${}_2F_1$ is the Gauss hypergeometric function, the most beautiful function there is. The vacuum block is obtained by setting $h = 0$, and we find the following singular behavior:[14]

$$G_\alpha(z) \approx |z|^{(\alpha-1)\Delta_L} \left| \frac{1-z^\alpha}{\alpha} \right|^{-2\Delta_L} = |1-z|^{-2\Delta_L} \left(1 + \frac{\Delta_L \Delta_H}{c} |1-z|^2 + O(|1-z|^4) \right). \tag{4.12}$$

Thus we see explicitly that the presence of the heavy state modifies the two-point function of the light field by introducing extra subleading divergences as $z \longrightarrow 1$.[15]

Naturally, less can be gleaned from the s-channel OPE. The expansion is significantly more complicated, in part because we do not know the spectrum or the OPE coefficients. Nevertheless, crossing tells us that the CFT must arrange itself in such a way that the asymptotic behavior of (4.4) matches the divergence structure of the vacuum block.

FIN.

[14]This cannot be the full four-point function—it has the wrong analytic structure away from $z = 1$—but it is accurate enough for us to find the corrections to $\langle \mathcal{O}_L(1) \mathcal{O}_L(z) \rangle$ incurred by the presence of $\mathcal{O}_H$.

[15]Remarkably, this result can be reproduced in the bulk by computing the length of a geodesic anchored at the two boundary points 1 and z. The bulk geometry changes in response to the heavy object created by $\mathcal{O}_H$, and the way this affects the geodesic length precisely reflects the CFT calculation above.

References

[1] S. Ribault, "Conformal field theory on the plane," [arXiv:1406.4290 [hep-th]].

[2] S. Ribault, "Minimal lectures on two-dimensional conformal field theory," SciPost Phys. Lect. Notes **1**, 1 (2018) doi:10.21468/SciPostPhysLectNotes.1 [arXiv:1609.09523 [hep-th]].

[3] P. Di Francesco, P. Mathieu and D. Senechal, "Conformal Field Theory," Springer-Verlag, 1997, ISBN 978-0-387-94785-3, 978-1-4612-7475-9 doi:10.1007/978-1-4612-2256-9.

[4] P. H. Ginsparg, "Applied Conformal Field Theory," [arXiv:hep-th/9108028 [hep-th]].

[5] X. Yin, "Aspects of Two-Dimensional Conformal Field Theories," PoS **TASI 2017**, 003 (2017) doi:10.22323/1.305.0003. https://pos.sissa.it/305/003/pdf.